ON THE PROPAGATION OF ELASTIC-PLASTIC WAVES DURING HEATING OF A SEMI-INFINITE BAR

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The propagation of elastic-plastic waves in a semi-infinite bar is considered when its unloaded end is heated in a linear manner. The choice of a linear variation of temperature makes it possible in this case to obtain an exact solution of a nonlinear heat conduction equation [1 and 2] and of the problem of propagation of stress waves. In doing so it has been possible to explain the qualitative peculiarities of solution of this type of problems. It has been ascertained that if the (constant) velocity of propagation of heat is equal to the velocity of propagation of elastic or of plastic disturbances (in the case of linear strain hardening), a "resonance" occurs and waves of strong discontinuity form.

For any rate of rise of temperature at the free end of the bar, a wave of unloading occurs. In contrast to the case of the usual wave of unloading [3], the speed of the unloading wave approaches the speed of propagation of plastic waves, not the speed of sound [4], or else coincides with the front of the thermal wave. This is explained by the fact that as temperature increases with a non-decreasing rate at the end of the bar, the wave of unloading is of an "internal" nature. If the wave of unloading [3] arises as a result of changes of the boundary regime, the occurance of the thermal wave of unloading is a consequence of the change of slope of the stress-strain diagram (for a purely elastic diagram and the same boundary conditions unloading does not take place).

For a certain relation among the parameters of the problem it is not possible to determine the equation of the wave of unloading from only the boundary conditions and the conditions of continuity of the solution together with its first derivatives everywhere. In order to find the velocity of the wave of unloading it is necessary to introduce additional considerations. As in [5], it is assumed that the material of the bar exhibits linear strain hardening. The mechanical characteristics are considered not to depend upon temperature.

1. The basic system of dimensionless Equations using the notations as in [5], may be written in the form

$$\frac{\partial T}{\partial \tau} = \frac{\partial^2 T^2}{\partial y^2} \quad \text{(equation of heat conduction)} \tag{1.1}$$

$$\frac{\partial^2 u}{\partial \tau^2} = \frac{\partial s}{\partial y} \quad \text{(equation of motion)} \tag{1.2}$$

$$s = \frac{\partial u}{\partial y} - T \text{ (equation of state)} \tag{1.3}$$

The initial and boundary conditions are

$$u(y, 0) = \partial u(y, 0) / \partial\tau = 0, \qquad s(0, \tau) = 0, \qquad T(y, 0) = 0 \tag{1.4}$$

Equation (1.1) admits exact solution

$$T = \begin{cases} {}^1/_2\beta^2\tau \quad - {}^1/_2\beta y & (y \leqslant \beta\tau) \\ 0 & (y \geqslant \beta\tau) \end{cases} \tag{1.5}$$

Here the last condition of (1.4) is obviously fulfilled and $T(0,\tau) = \frac{1}{2}\beta^2\tau$ at the free end of the bar.

Let us consider the case for $\beta > 1$. The solution of the systems (1.2) and (1.3), taking account of (1.4), has the form (Fig.1)

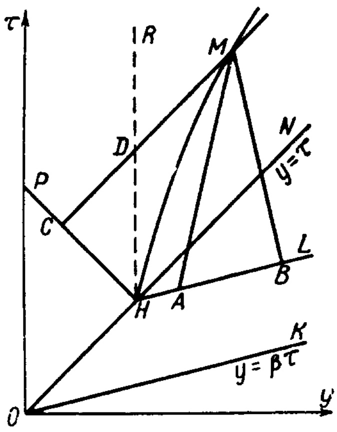

Fig.1

in the region yOK

$$u = 0$$

in the region KON

$$u = \frac{\beta^3}{4(\beta^2-1)}(y^2+\tau^2) - \frac{\beta^2}{2(\beta^2-1)}y\tau - \frac{\beta}{4}y^2$$

$$s = \frac{\beta^3}{2(\beta^2-1)}(y-\beta\tau) \tag{1.6}$$

in the region $NO\tau$

$$u = -\frac{\beta^3}{4(\beta+1)}(y^2+\tau^2) + \frac{\beta^2}{2}y\tau - \frac{\beta}{4}y^2$$

$$s = -\frac{\beta^3}{2(\beta+1)}y \tag{1.7}$$

It is apparent from (1.6) that the straight lines $y = \beta\tau + \text{const}$ in the region KON are lines of constants of displacement, velocity and stress.

Along HL (1.8)

$$y = \beta\tau + \frac{2(1-\beta^2)}{\beta^3}, \quad u = \frac{\beta^2-1}{\beta^3}, \quad u_\tau = \frac{1}{\beta}, \quad u_y = -\frac{1}{\beta^2}, \quad s = -1$$

(subscripts τ and y denote derivatives). Therefore solution (1.6) is valid only in the region $KOHL$; there is a plastic zone above HL.

Solution (1.7) is valid in the region OHP. By direct verification we can assure ourselves that above PH there exists no solution corresponding to loading. Therefore, a wave of unloading with the positive inclination HM begins at the point $H\,(2(1+\beta)/\beta^3,\ 2(1+\beta)/\beta^3)$.

When there is linear strain hardening we have in the plastic region MHL, during loading, (Fig. 2)

$$u_y = e \equiv e^0 + T, \qquad s = q^2 e^0 + q^2 - 1$$
$$q^2 = E_1 / E$$

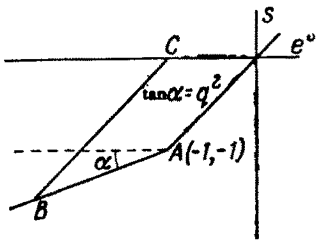

Fig.2

where E_1 is the modulus in the hardening range. From this

$$s = q^2 u_y - q^2 T + q^2 - 1 \qquad (1.9)$$

From (1.2), (1.5), and (1.9) we obtain Equation of motion in terms of displacements

$$u_{\tau\tau} = q^2 u_{yy} + {}^1/_2\, q^2 \beta$$

The characteristics of last Equation and the relations on them may be written

$$dy = + q d\tau, \qquad du_\tau = + q\, du_y + {}^1/_2\, q\beta\, dy$$

or, after integration

$$u_\tau = + q u_y + {}^1/_2\, q\beta y + C_{1,2} \qquad (1.10)$$

We denote Equation of the wave of unloading by $\tau = \omega\,(y)$ We shall assume that its initial velocity satisfies inequality

$$1 < \omega'\,(2(1+\beta) / \beta^3) < 1 / q \qquad (1.11)$$

It is then possible to draw through an arbitrary point M of the wave of unloading which is sufficiently close to the point H the characteristic of positive slope MC and in the plastic zone the characteristics of both directions, MA and MB, which intersect the elastic-plastic boundary HL. The relations (1.10) hold along MA and MB. From this fact, taking conditions (1.8) into account, we obtain at the point $M\,(y, \omega\,(y))$

$$u_y = -\frac{q^2\beta^2}{2\,(\beta^2 - q^2)}\,\omega\,(y) + \frac{q^2\beta}{2\,(\beta^2 - q^2)}\,y - \frac{1 - q^2}{\beta^2 - q^2}, \qquad u_\tau = -\beta u_y \quad (1.12)$$

We introduce the function $f\,(y)$, which is equal to the stress in elements of the bar just behind the wave of unloading. Following Equation of state will then hold in the region RHM (BC, Fig. 2)

$$s = u_y - T + \frac{q^2 - 1}{q^2}\, f\,(y) + \frac{q^2 - 1}{q^2}$$

and Equation of motion may be written out in the form

$$\frac{\partial^2 u}{\partial \tau^2} = \frac{\partial^2 u}{\partial y^2} + \frac{q^2 - 1}{q^2}\, f'\,(y) + \frac{1}{2}\,\beta$$

Along the characteristic DM

$$du_\tau = du_y + \frac{q^2 - 1}{q^2}\, df\,(y) + \frac{1}{2}\,\beta\, dy$$

or, after integration

$$u_\tau = u_y + \frac{q^2 - 1}{q^2} f\,(y) + \frac{1}{2}\,\beta y + C_3 \qquad (1.13)$$

In the region τPHR Equations (1.2) and (1.3) apply, and we have along CD

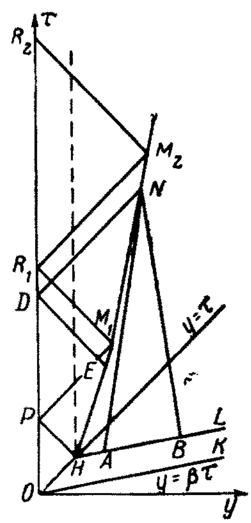

Fig. 3

$$u_\tau = u_y + \frac{1}{2}\beta y + C_4, \qquad C_4 = \frac{\beta^2(2\beta+1)}{2(\beta+1)}(y-\tau) \tag{1.14}$$

Obviously, $f(2(1+\beta)/\beta^2) = -1$. Requiring the continuity of u_τ and u_y at point D, we find by comparing (1.14) with (1.13) that

$$C_3 = C_4 + (q^2-1)\ /\ q^2$$

Thus, at the point $M(y, \omega(y))$ on the wave of unloading we have

$$u_\tau = u_y + \frac{q^2-1}{q^2} f(y) + \frac{1}{2}\beta y + \frac{\beta^2(2\beta+1)}{2(\beta+1)}[y - \omega(y)] + \frac{q^2-1}{q^2} \tag{1.15}$$

Moreover, $s(y, \omega(y)) = f(y)$, or

$$u_y = f(y)\ /\ q^2 + \beta^2\omega(y)/2 - \beta y\ /\ 2 + (1-q^2)\ /\ q^2 \tag{1.16}$$

Eliminating u_τ, u_y and $f(y)$, from (1.12),(1.15) and (1.16), we obtain Equation of the initial portion of the wave of unloading in the form of the straight line

$$\tau = \xi_1 y + \eta_1, \qquad \xi_1 = \frac{q^2+2\beta+1}{2q^2+q^2\beta+\beta}, \qquad \eta_1 = -\frac{2(1-q^2)(1+\beta)^2}{\beta^3(2q^2+q^2\beta+\beta)} \tag{1.17}$$

Here conditions (1.11) are satisfied for any $0 \leqslant q < 1$.

This solution is valid until the characteristics MC and PH intersect, i.e. up to the point M_1 (Fig. 3).

Analogously to the preceding, by considering the characteristics NA, NB, ND and DE we find that the next portion of the wave of unloading M_1M_2 is a segment of a straight line with slope ξ_2 greater than ξ_1,

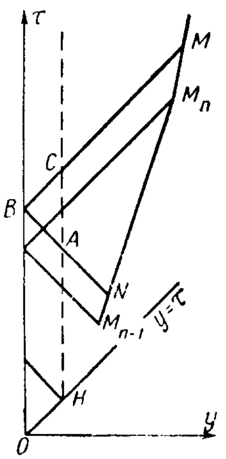

Fig.4

$$\tau = \xi_2 y + \eta_2$$

$$\xi_2 = \frac{q^4+6q^2+4q^2\beta+4\beta+1}{4q^4+q^4\beta+4q^2+6q^2\beta+\beta}$$

$$\eta_2 = -\frac{2(1-q^2)(-q^2+4q^2\beta+q^2\beta^2+\beta^2+4\beta+1)}{\beta^3(4q^4+q^4\beta+4q^2+6q^2\beta+\beta)}$$

where (1.18)

$$\xi_2 = \frac{\xi_1(1+q^2)+2}{2q^2\xi_1+q^2+1}, \qquad \eta_2 = \frac{(1-q^2)(\beta^2\eta_1-4\xi_1-4)}{\beta^2(2q^2\xi_1+q^2+1)}$$

Beyond point M_2 we obtain $\tau = \xi_3 y + \eta_3$, $\xi_3 > \xi_2$, etc. Thus the wave of unloading presents a broken line. The slopes and intercepts of the individual segments of it are related by recurrence Formulas (1.19)

$$\xi_{n+1} = \frac{\xi_n(1+q^2)+2}{2q^2\xi_n+q^2+1}, \qquad \eta_{n+1} = \frac{(1-q^2)(\beta^2\eta_n-4\xi_n-4)}{\beta^2(2q^2\xi_n+q^2+1)}$$

which we shall prove by the method of mathematical induction, taking (1.18) in consideration. Let us consider an arbitrary point $M\ (y, \tau)$ on the $(n+1)$th portion of the unloading wave (Fig.4), and the characteristics MB and BN. At the point

$$N: \left(\frac{\tau - y}{\xi_n + 1} - \frac{\eta_n}{\xi_n + 1},\quad \frac{\xi_n (\tau - y)}{\xi_n + 1} + \frac{\eta_n}{\xi_n + 1} \right)$$

we have from (1.12)

$$u_y = \frac{q^2\beta (1 - \beta\xi_n)(\tau - y)}{2(\beta^2 - q^2)(\xi_n + 1)} - \frac{q^2\beta(1+\beta)\eta_n}{2(\beta^2 - q^2)(\xi_n + 1)} - \frac{1 - q^2}{\beta^2 - q^2}, \quad u_\tau = -\beta u_y$$

Along NA

$$u_\tau = -u_y - \beta y / 2 + (1 - q^2) f(y) / q^2 + C_1$$

$$C_1 = \frac{\beta^2 [(\beta^2 - q^2\beta^2 + q^2\beta - q^2)\xi_n + q^2(\beta - 1)]}{2(\beta^2 - q^2)(\xi_n + 1)} (\tau - y) + \\ + \frac{(1 - q^2)\beta^4\eta_n}{2(\beta^2 - q^2)(\xi_n + 1)} + \frac{(1 - q^2)(\beta^2 - q^2\beta^2 + q^2\beta - q^2)}{q^2(\beta^2 - q^2)}$$

Along AB

$$u_\tau = -u_y - {}^1/_2\beta y + C_2, \quad C_2 = C_1 - (1 - q^2) / q^2$$

At point B, taking into consideration the boundary condition $s(0, \tau) = 0$, we have

$$u_y = {}^1/_2\,\beta^2 (\tau - y), \qquad u_\tau = -{}^1/_2\,\beta^2(\tau - y) + C_2$$

Further, along BC

$$u_\tau = u_y + {}^1/_2\,\beta y + C_3, \qquad C_3 = -\beta^2(\tau - y) + C_2$$

along CM

$$u_\tau = u_y + {}^1/_2\,\beta y + \frac{q^2 - 1}{q^2} f(y) + C_4, \qquad C_4 = C_3 + \frac{q^2 - 1}{q^2} \tag{1.20}$$

Combining (1.20) with the relations (1.12) and (1.16) which are valid at the point M, provided that

$$1 < \xi_{n+1} < 1/q \tag{1.21}$$

and eliminating u_y, u_τ and $f(y)$, we obtain

$$\tau = \frac{\xi_n(1 + q^2) + 2}{2q^2\xi_n + q^2 + 1} y + \frac{(1 - q^2)(\beta^2\eta_n - 4\xi_n - 4)}{\beta^2(2q^2\xi_n + q^2 + 1)}$$

Formulas (1.19) follow from this. Conditions (1.21) are fulfilled for any n and $0 \leqslant q < 1$.

The slopes ξ_n form an increasing sequence with limit equal to $\xi_0 = 1/q$. Equation of the asymptote of the wave of unloading has the form

$$\tau = \frac{1}{q} y - \frac{4(1 - q^2)}{q\beta^2(q\beta^2 + q - \beta^2 + 1)} \tag{1.22}$$

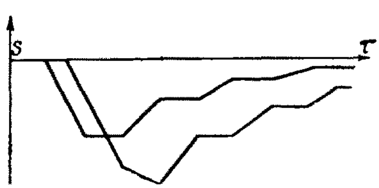

Fig.5

The graphs of the variation of stress with time at sections of the bar, are, of course, also broken lines. Their form is shown schematically in Fig.5. After the wave of unloading passes, the stress begins to drop off, approaching zero asymptotically, but remaining constant with time in the characteristic triangles PM_1R_1, $R_1M_2R_2$ etc., which border on the axis $O\tau$ (Fig. 3).

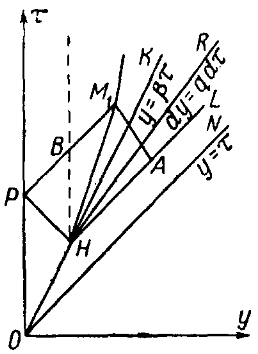

Fig.6

For $q \to 1$ the solution which has been constructed reverts to solution (1.6) and (1.7). In this case unloading does not occur and the wave of unloading $\tau = \omega(y)$ becomes the "wave of neutral loading" $\tau = y$, on which the stress at any section of the bar attains a maximum (in modulus) and subsequently remains constant.

2. If we pass to the limit $\beta \to 1$ in the solution which has been obtained, i.e. if the velocity of propagation of heat in the bar becomes equal to the velocity of propagation of elastic disturbances, then a "resonance" occurs and the straight line $\tau = y$ will be a wave of strong discontinuity. The stress discontinuity is equal to

$$[s] \equiv s(y, y+0) - s(y, y-0) = \begin{cases} -{}^1/_4 y & (y \leqslant 4) \\ -1 & (y \geqslant 4) \end{cases} \tag{2.1}$$

The case $\beta < 1$. Taking into consideration the initial and boundary conditions (1.4), we have a solution which, together with its first derivatives, is everywhere continuous (Fig.6)

region yON

$$u = 0$$

region NOK

$$u = \frac{\beta^4}{4(1-\beta^2)}(y-\tau)^2, \qquad s = \frac{\beta^4}{2(1-\beta^2)}(y-\tau) \tag{2.2}$$

region $KO\tau$

$$u = -\frac{\beta^3}{4(1+\beta)}(y^2+\tau^2) + \frac{1}{2}\beta^2 y\tau - \frac{1}{4}\beta y^2, \qquad s = -\frac{\beta^3 y}{2(1+\beta)} \tag{2.3}$$

We note that Formulas (2.3) coincide with (1.7). Solution (2.2) is valid only in the region $NOHL$. Along HL

$$y = \tau - 2(1-\beta^2)/\beta^4, \quad u = (1-\beta^2)/\beta^4, \; u_\tau = 1, \quad u_y = s = -1 \tag{2.4}$$

There is a plastic zone above HL.

Solution (2.3) is valid in the triangle OHP. Along HP

$$y = -\tau + 2(1+\beta)^2/\beta^4 \tag{2.5}$$

$$u = -\frac{\beta(4\beta^2+3\beta+1)}{4(1+\beta)}\tau^2 + \frac{(1+\beta)(2\beta^2+\beta+1)}{\beta^3}\tau - \frac{(1+\beta)^3(\beta^2+\beta+1)}{\beta^7}$$

$$u_\tau = -\frac{\beta^2(2\beta+1)}{2(\beta+1)}\tau + \frac{(1+\beta)^2}{\beta^2}, \qquad u_y = \frac{\beta(2\beta^2+2\beta+1)}{2(\beta+1)}\tau - \frac{(1+\beta)(\beta^2+\beta+1)}{\beta^3}$$

We construct, analogously to the previous case, the equation of the wave of unloading, which begins at the point $H(2(1+\beta)/\beta^3,\ 2(1+\beta)/\beta^4)$ in the form of a broken line.

Equation of the first section of it, HM_1, is

$$\tau = \xi_1{}^0 y + \eta_1{}^0, \quad \xi_1{}^0 = \frac{q^2+2\beta+1}{2q^2+q^2\beta+\beta}, \qquad \eta_1{}^0 = -\frac{4(\beta^2-q^2)(1+\beta)}{\beta^4(2q^2+q^2\beta+\beta)} \tag{2.6}$$

where

$$f(y) = -\frac{q^2\beta^3}{2q^2+q^2\beta+\beta}\,y - \frac{\beta(1-q^2)}{2q^2+q^2\beta+\beta} \tag{2.7}$$

Recurrence Formulas, analogous to (1.19), have the form

$$\xi_{n+1}^{\ 0} = \frac{\xi_n{}^0(1+q^2)+2}{2q^2\xi_n{}^0+q^2+1}, \qquad \eta_{n+1}^{\ 0} = \frac{\beta^4(1-q^2)\,\eta_n{}^0 - 4(\beta^2-q^2)(\xi_n{}^0+1)}{\beta^4(2q^2\xi_n{}^0+q^2+1)} \tag{2.8}$$

The sequence of slopes $\{\xi_n^0\}$ coincides with the sequence $\{\xi_n\}$. Equation of the asymptote of the wave of unloading is

$$\tau = y/q - 2(\beta^2-q^2)/q^2\beta^4 \tag{2.9}$$

Formulas (2.6) to (2.9) which have been obtained are valid under the assumption that $1/\beta < \xi_1 < 1/q$, i.e. for $0 \leqslant q < \beta$. Passing to the limit as $q \to \beta$, we find that the wave of unloading coincides with the thermal wave $y = \beta\tau$ in this case. The stress just behind the wave front equals

$$s = -\beta^4 y/(1+\beta)^2 - (1-\beta)/(1+\beta)$$

from (2.7).

On the other hand, in the region LHK (Fig.6), as easily verified, solution which satisfies the conditions of continuity along the elastic-plastic boundary HL has the form

$$u = \tau - y - (1-\beta^2)/\beta^4, \qquad s = -1$$

Therefore, for $q = \beta$, the thermal wave is simultaneously the unloading wave and a wave of strong discontinuity. The stress discontinuity is equal to

$$[s] = \begin{cases} -\beta^4 y/(1+\beta)^2 + 2\beta/(1+\beta) & (y \geqslant 2(1+\beta)/\beta^3) \\ 0 & (y \leqslant 2(1+\beta)/\beta^3) \end{cases} \tag{2.10}$$

3. Now let $\beta < q$. It is natural to look for the equation of the initial portion of the wave of unloading in the form of the straight line

$$\tau = \omega(y) = \xi y + 2(1+\beta)(1-\beta\xi)/\beta^4 \tag{3.1}$$

with unknown slope ξ. Let us assume that $\xi > 1/\beta$. Then it is only possible to draw one characteristic ML in the plastic region from an arbitrary point M of the unloading wave which intersects the elastic-plastic boundary HL (Fig.6). It is, therefore, necessary to solve the problem in terms of the displacements.

Taking (2.4) into account, we write out solution in the region LHN in the form

$$u = \tau - y - (1-\beta^2)/\beta^4, \quad u_y = -1, u_\tau = 1$$

By requiring the continuity of u, u_y and u_τ along HR and HK, we have in the regions RHK and KHM, respectively

$$u = \frac{1-q}{2q}(y+q\tau) + \varphi(y-q\tau) - \varphi\left(\frac{2(1+\beta)(\beta-q)}{\beta^4}\right) - \frac{(1+\beta)(\beta-q^2)}{q\beta^4} \tag{3.2}$$

$$u = \frac{\beta^2}{8(\beta+q)}(y+q\tau)^2 + \frac{\beta^2}{8(\beta-q)}(y-q\tau)^2 - \frac{1}{4}\beta y^2 + \tag{3.3}$$
$$+ \frac{1-q}{2q}(y+q\tau) + \varphi(y-q\tau) - \varphi\left(\frac{2(1+\beta)(\beta-q)}{\beta^4}\right) - \frac{(1+\beta)(\beta-q^2)}{q\beta^4}$$

The function φ is found from the obvious relation $s = f(y)$, on the wave of unloading, i.e.

$$u_y(y, \omega(y)) = \frac{1}{q^2} f(y) - \frac{1}{2}\beta(1-\beta\xi)y + \frac{(1+\beta)(1-\beta\xi)}{\beta^2} + \frac{1-q^2}{q^2}$$

From this,

$$\varphi(x) = \frac{1-q\xi}{q^2}\int_{\alpha}^{A_1(x)} f(x)\,dx - \frac{\beta^3(1-\beta\xi)}{4(\beta^2-q^2)(1-q\xi)}x^2 +$$
$$+ \frac{(1+\beta)(1-\beta\xi)}{\beta(\beta+q)(1-q\xi)}x + \frac{(1-q)(2+q)}{2q^2}x + D$$

$$A_1(x) = \frac{x}{1-q\xi} + \frac{2q(1+\beta)(1-\beta\xi)}{\beta^4(1-q\xi)}, \qquad \alpha = \frac{2(1+\beta)}{\beta^3} \tag{3.4}$$

We obtain Cauchy problems for the regions MHB and BHP successively, and write out solutions in these regions in the following manner:

region MHB

$$u = \frac{1-q^2}{q^2}F_1(y) + \frac{(q-1)(1+\xi)}{2q}F_2(y+\tau) + \frac{(q+1)(1-\xi)}{2q}F_3(y-\tau) +$$
$$+ \frac{\xi\beta^3(1+q\xi)}{4(\beta+q)(1-\xi^2)}(y^2+\tau^2) + \frac{\beta^2(q-q\beta\xi-q\xi^2-\beta\xi^2)}{2(\beta+q)(1-\xi^2)}y\tau - \frac{1}{4}\beta y^2 +$$
$$+ \frac{(1+\beta)(1-\beta\xi)(1+q\xi)}{\beta(\beta+q)(1-\xi^2)}y + \frac{1-q^2}{q^2}y - \frac{(1+\beta)(1-\beta\xi)(q+\xi)}{\beta(\beta+q)(1-\xi^2)}\tau -$$
$$- \frac{1-q}{q}\tau + \frac{(1+\beta)^2(1-\beta\xi)(q-\beta+\xi-q\beta\xi)}{\beta^5(\beta+q)(1-\xi^2)} - \frac{(1+\beta)(2\beta+q^2-2q-q^2\beta)}{q^2\beta^4} \tag{3.5}$$

region BHP

$$u = \frac{(q-1)(1+\xi)}{2q}F_2(y+\tau) + \frac{(q+1)(1-\xi)}{2q}F_3(y-\tau) - \frac{1}{4}\beta y^2 +$$
$$+ \frac{\xi\beta^3(1-q\xi)}{4(\beta+q)(1-\xi^2)}(y^2+\tau^2) + \frac{\beta^2(q-q\beta\xi-q\xi^2-\beta\xi^2)}{2(\beta+q)(1-\xi^2)}y\tau +$$
$$+ \frac{(1+\beta)(1-\beta\xi)(1+q\xi)}{\beta(\beta+q)(1-\xi^2)}y - \frac{(1+\beta)(1-\beta\xi)(q+\xi)}{\beta(\beta+q)(1-\xi^2)}\tau - \frac{1-q}{q}\tau +$$
$$+ \frac{(1+\beta)^2(1-\beta\xi)(q-\beta+\xi-q\beta\xi)}{\beta^5(\beta+q)(1-\xi^2)} + \frac{(1+\beta)(2-q-q\beta)}{q\beta^4} \tag{3.6}$$

Here

$$F_1(y) = \int_\alpha^y f(x)\,dx, \qquad F_2(y+\tau) = \int_\alpha^{A_2} f(x)\,dx, \qquad F_3(y-\tau) = \int_\alpha^{A_3} f(x)\,dx$$

$$A_2 = \frac{y+\tau}{1+\xi} - \frac{2(1+\beta)(1-\beta\xi)}{\beta^4(1+\xi)}, \qquad A_3 = \frac{y-\tau}{1-\xi} + \frac{2(1+\beta)(1-\beta\xi)}{\beta^4(1-\xi)} \qquad (3.7)$$

By requiring the continuity of u_y along the characteristic HP, we find the function $f(y)$ from (1.3), (1.5), (2.5), (3.6) and (3.7)

$$f(y) = \frac{q\beta^3(\beta\xi - q\beta\xi - 2\beta - q - 1)}{2(\beta+q)(1+\beta)(1+q)}\, y - \frac{\beta(1-q)(1+q\xi)}{(\beta+q)(1+q)} \qquad (3.8)$$

Here u and u_τ turn out to be continuous along HP for any ξ.

Solving the problem in the region $MP\tau$, it is easy to ascertain, that boundary condition $s(0, \tau) = 0$ satisfies for any ξ.

To determine the equation of the wave of unloading, it is necessary, in addition, to verify the conditions of loading and unloading at an arbitrary section of the bar before and after passage of the wave of unloading, i.e. (considering that the stress is everywhere negative)

$$\frac{\partial s(y,\tau)}{\partial\tau} \leqslant 0 \quad (\tau \leqslant \omega(y)), \qquad \frac{\partial s(y,\tau)}{\partial\tau} > 0 \quad (\tau > \omega(y)) \qquad (3.9)$$

We find from (3.3), (3.4), (3.8) and (1.9) that the first of these conditions cannot be satisfied for any $\xi > 1/\beta$. Assuming that the wave of unloading lies below the thermal wave, i.e. $\xi < 1/\beta$, we convince ourselves in like manner that second condition of (3.9) is violated.

Thus, the velocity of the wave of unloading cannot be greater than the velocity of propagation of heat in the bar, and for any $\beta < q$ the wave of unloading coincides with the front of the thermal wave. Here all the conditions of continuity, the boundary conditions and conditions (3.9) are satisfied.

For $q \to 1$ the solution reverts to (2.2) and (2.3). Unloading does not occur in this case, and the broken line $\tau = \omega(y)$ becomes the "straight line of neutral loading" $\tau = y$.

Of all the mechanical characteristics of a material the one which depends most on temperature is the elastic limit. We remark that the solution which has been obtained is valid for any temperature dependence of the elastic limit, since the elastic-plastic boundary HL (Fig. 1) is a line of constant temperature in the case $\beta > 1$ and in the case $\beta \leqslant 1$ the temperature is zero along the elastic-plastic boundary (Fig. 6).

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